Programme 5

Z transforms

Frames 1 to 43

Learning outcomes

When you have completed this Programme you will be able to:

- Define the Z transform of a sequence and derive transforms of specified sequences
- Make reference to a table of standard Z transforms
- Recognise the Z transform as being a linear transform and so obtain the transform of linear combinations of standard sequences
- Apply the first and second shift theorems, the translation theorem, the initial and final value theorems and the derivative theorem
- Use partial fractions to derive the inverse transforms
- Solve linear, first-order, constant coefficient recurrence relations
- Demonstrate the relationship between the Laplace transform and the Z transform

Introduction

1

The Laplace transform deals with continuous functions and can be used to solve many differential equations that arise in science and engineering. There are occasions, however, when we have to deal with discrete functions – *sequences* – and their associated **difference equations**. For example, the central processing unit of your computer can only handle information in the form of pulses of electricity. This information transmission is called **digital** transmission. There are, however, times when information is fed into the computer in the form of a continuously varying signal called an **analogue** signal. For instance, a mouse can be moved about the flat surface of your desk in a continuous manner but the central processing unit will only recognise position on the screen to the nearest pixel. The analogue signal coming from the mouse needs to be converted into a digital signal for recognition by the computer's central processing unit. This conversion of a signal from analogue to digital is achieved by a device called a **demodulator** that *samples* the analogue signal at regular intervals of time and outputs the sampled values as the digital signal – as a sequence of values. The Z transform, which is allied to the Laplace transform, deals with such sequences and the recurrence relations – or difference equations – that arise.

Sequences

2

The sequence $\ldots, 3^{-2}, 3^{-1}, 3^0, 3, 3^2, 3^3, \ldots$ has a general term of the form 3^k and as a shorthand notation we use $\{3^k\}_{-\infty}^{\infty}$ to represent this sequence and to indicate that the powers range from $-\infty$ to ∞. The sum

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \ldots + \left(\frac{3}{z}\right)^{-1} + \left(\frac{3}{z}\right)^0 + \left(\frac{3}{z}\right)^1 + \left(\frac{3}{z}\right)^2 + \ldots$$

is called the ***Z* transform** of the sequence, $Z\{3^k\}_{-\infty}^{\infty}$, and is denoted by $F(z)$, where the complex number z is chosen to ensure that the sum is finite. We say that

$$\{3^k\}_{-\infty}^{\infty} \text{ and } Z\{3^k\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k \text{ form a } Z \textit{ transform pair.}$$

▶

For our purposes we shall consider only *causal sequences* of the form $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$ which for brevity we shall denote by $\{x_k\}$ with corresponding Z transform

$$Z\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k}.$$

Notice that this is the *definition* of the Z transform of the sequence $\{x_k\}$. For example, the *unit impulse* sequence $\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$ has the Z transform

$Z\{\delta_k\} = \ldots\ldots\ldots\ldots$ valid for $\ldots\ldots\ldots\ldots$ values of z

3

$$\boxed{Z\{\delta_k\} = 1 \text{ valid for all values of } z}$$

Because

$$Z\{\delta_k\} = \sum_{k=0}^{\infty} \frac{\delta_k}{z_k}$$
$$= 1 + \frac{0}{z} + \frac{0}{z^2} + \ldots = 1$$

Try another.

The sequence $\{u_k\} = \{1, 1, 1, \ldots\} = \{1\}$ is called the *unit step* sequence and has the Z transform

$\ldots\ldots\ldots\ldots$ provided $|z| \ldots\ldots\ldots\ldots$

Next frame

4

$$\boxed{\frac{z}{z-1} \text{ provided } |z| > 1}$$

Because

$$Z\{u_k\} = F(z)$$
$$= \sum_{k=0}^{\infty} \frac{u_k}{z^k} = \sum_{k=0}^{\infty} \frac{1}{z^k}$$
$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ldots$$

Comparing this to the series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$F(z) = \frac{1}{1 - \dfrac{1}{z}} \text{ provided } \left|\frac{1}{z}\right| < 1$$
$$= \frac{z}{z-1} \text{ provided } |z| > 1$$

And another.

Given the causal sequence $\{x_k\} = \{1, a, a^2, a^3, a^4, \ldots\} = \{a^k\}$

the Z transform is $\ldots\ldots\ldots\ldots$

Next frame

5

$$\boxed{\frac{z}{z-a} \text{ provided } |z| > a}$$

Because

$$\begin{aligned} Z\{a^k\} &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\ &= \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k \\ &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \end{aligned}$$

Comparing this to the series expansion of $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots \\ &= \frac{1}{1 - \dfrac{a}{z}} \text{ provided } \left|\frac{a}{z}\right| < 1. \end{aligned}$$

That is, multiplying numerator and denominator by z

$$F(z) = \frac{z}{z-a} \text{ provided } |z| > |a|$$

Therefore $\{a^k\}$ and $F(z) = \dfrac{z}{z-a}$, $(|z| > |a|)$ form a Z transform pair.

Let's try another. The sequence $\{x_k\} = \{0, 1, 2, 3, 4, \dots\} = \{k\}$ has the Z transform

$$Z\{k\} = F(z) = \dots\dots\dots\dots$$

Answer in the next frame

6

$$\boxed{F(z) = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots}$$

Because

$$\begin{aligned} Z\{k\} &= F(z) \\ &= \sum_{k=0}^{\infty} \frac{x_k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\ &= 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots \end{aligned}$$

By comparing this sequence with the derivative of $(1-x)^{-1}$ and its series representation, this sequence can be written as a rational expression in z as $F(z) = \dots\dots\dots\dots$

7

$$\boxed{F(z) = \frac{z}{(z-1)^2}}$$

Because

$$F(z) = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \dots$$

Comparing this with the series expansion

$$\begin{aligned} 1 + 2x + 3x^2 + 4x^3 + \cdots &= \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + \cdots) \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(1-x)^{-1} = \frac{1}{(1-x)^2} \end{aligned}$$

then we can see that by multiplying $F(z)$ by z

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots = \frac{1}{(1-1/z)^2}$$

so, dividing both sides by z gives

$$F(z) = \frac{1}{z(1-1/z)^2} = \frac{z}{(z-1)^2}$$

Next frame

Table of *Z* transforms

8

We list the results that we have obtained so far as well as some additional ones for future reference.

Sequence	**Transform $F(z)$**	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \dots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, \dots\}$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \dots\}$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \dots\}$	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \dots\}$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \dots\}$	$\dfrac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \dots\}$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

Next frame

Properties of *Z* transforms

1 *Linearity*

9

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\} + b\{y_k\}) = aZ\{x_k\} + bZ\{y_k\}$$

For example, the Z transform of the sequence $\{k\}$ is $Z\{k\} = \ldots\ldots\ldots\ldots$ and the Z transform of the sequence $\{e^{-2k}\}$ is $Z\{e^{-2k}\} = \ldots\ldots\ldots\ldots$

10

$$\boxed{Z\{k\} = \frac{z}{(z-1)^2} \text{ and } Z\{e^{-2k}\} = \frac{z}{z-e^{-2}}}$$

Because

$Z\{k\} = \dfrac{z}{(z-1)^2}$ from the table and, also from the table,

$Z\{a^k\} = \dfrac{z}{z-a}$ so when $a = e^{-2}$,

$$Z\{e^{-2k}\} = \frac{z}{z-e^{-2}}$$

Consequently, the Z transform of $3\{k\} - 5\{e^{-2k}\}$ is $\ldots\ldots\ldots\ldots$

11

$$\boxed{\frac{-5z^3 + 13z^2 - z(3e^{-2} + 5)}{(z-1)^2(z-e^{-2})}}$$

Because

$$\begin{aligned} Z(3\{k\} - 5\{e^{-2k}\}) &= 3Z\{k\} - 5Z\{e^{-2k}\} \\ &= \frac{3z}{(z-1)^2} - \frac{5z}{(z-e^{-2})} \\ &= \frac{3z(z-e^{-2}) - 5z(z-1)^2}{(z-1)^2(z-e^{-2})} \\ &\frac{3z^2 - 3ze^{-2} - 5z^3 + 10z^2 - 5z}{(z-1)^2(z-e^{-2})} \\ &= \frac{-5z^3 + 13z^2 - z(3e^{-2} + 5)}{(z-1)^2(z-e^{-2})} \end{aligned}$$

▶

2 ***First shift theorem*** (shifting to the left)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

is the Z transform of the sequence that has been shifted by m places to the left. For example

$$Z\{x_{k+1}\} = zF(z) - zx_0$$
$$Z\{x_{k+2}\} = z^2F(z) - z^2x_0 - zx_1$$

These will be used later when solving difference equations. Note the similarity between these results and the Laplace transforms for the first and second derivatives for continuous functions.

For example, given that $Z\{4^k\} = \dfrac{z}{z-4}$ then

$$Z\{4^{k+3}\} = \ldots\ldots\ldots\ldots$$

12

$$\boxed{\dfrac{64z}{z-4}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1}x_1 + \ldots + zx_{m-1}\right]$$

so

$$\begin{aligned}
Z\{4^{k+3}\} &= z^3 Z\{4^k\} - \left[z^3 4^0 + z^2 4^1 + z4^2\right] \text{ where } Z\{4^k\} = \frac{z}{z-4}\\
&= z^3\frac{z}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4}{z-4} - \left[z^3 + 4z^2 + 16z\right]\\
&= \frac{z^4 - (z^3 + 4z^2 + 16z)(z-4)}{z-4}\\
&= \frac{z^4 - (z^4 - 64z)}{z-4}\\
&= \frac{64z}{z-4}
\end{aligned}$$

In this way we have derived the Z transform of the sequence $\{64, 256, 1024, \ldots\}$ by shifting the sequence $\{1, 4, 16, 64, 256, \ldots\}$ three places to the left and losing the first three terms.

Try another. Given that $Z\{k\} = \dfrac{z}{(z-1)^2}$ then

$$Z\{(k+1)\} = \ldots\ldots\ldots\ldots$$

13

$$\boxed{\frac{z^2}{(z-1)^2}}$$

Because

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

so

$$Z\{k+1\} = z\frac{z}{(z-1)^2} - [z \times 0]$$
$$= \frac{z^2}{(z-1)^2}$$

3 ***Second shift theorem*** (shifting to the right)

If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m} F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

For example, given that $Z\{x_k\} = \dfrac{z}{z-1}$ then

$$Z\{x_{k-3}\} = \ldots\ldots\ldots\ldots$$

14

$$\boxed{\frac{1}{z^2(z-1)}}$$

Because

$$Z\{x_{k-m}\} = z^{-m} F(z)$$

so

$$Z\{x_{k-3}\} = z^{-3}\frac{z}{z-1}$$
$$= \frac{1}{z^2(z-1)}$$

In this way we have derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, \ldots\}$ by shifting the sequence $\{1, 1, 1, 1, \ldots\}$ three places to the right and defining the first three terms as zeros.

Try this one. The sequence $\{x_k\}$ with Z transform

$$Z\{x_k\} = \frac{1}{(z-a)}, \text{ where } a \text{ is a constant, is } \{\ldots\ldots\ldots\ldots\}$$

15

$$\boxed{\{a^{k-1}\}}$$

Because

From the table of transforms the nearest transform to the one in question is $\dfrac{z}{(z-a)}$ which is the Z transform of$\{a^k\}$. Now

$$\begin{aligned}\frac{1}{(z-a)} &= \frac{1}{z} \times \frac{z}{(z-a)}\\ &= z^{-1}F(z) \quad \text{where } F(z) = Z\{a^k\}\end{aligned}$$

and so

$$\frac{1}{(z-a)} = Z\{a^{k-1}\}$$

which is the Z transform of $\{a^k\}$, shifted one place to the right.

4 Translation

If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

For example, $Z\{k\} = \dfrac{z}{(z-1)^2}$ so that $Z\{2^k k\} = \dots\dots\dots\dots$

16

$$\boxed{\frac{2z}{(z-2)^2}}$$

Because

Since $Z\{k\} = \dfrac{z}{(z-1)^2} = F(z)$ then by the translation property

$$\begin{aligned}Z\{2^k k\} &= F(2^{-1}z)\\ &= \frac{2^{-1}z}{(2^{-1}z-1)^2}\\ &= \frac{2z}{(z-2)^2}\end{aligned}$$

▶

5 *Final value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

For example, the sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}.$$

Now

$$\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \underset{z\to 1}{Lim}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

and

$$\underset{k\to\infty}{Lim}\left\{\left(\frac{1}{2}\right)^k\right\} = 0 \text{ which confirms the final value theorem.}$$

Using the final value theorem the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2+2z}{(z-1)(5z-1)^2} \text{ is } \ldots\ldots\ldots\ldots$$

17

$\boxed{0{\cdot}75}$

Because

$$\begin{aligned}\underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} &= \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\}\\ &= \underset{z\to 1}{Lim}\left\{\frac{10z+2}{(5z-1)^2}\right\}\\ &= \frac{12}{16}\\ &= 0{\cdot}75\end{aligned}$$

6 *The initial value theorem*

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\{F(z)\}$$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and $\underset{z\to\infty}{Lim}\, F(z) = \underset{z\to\infty}{Lim}\dfrac{z}{z-a} = \underset{z\to\infty}{Lim}\dfrac{1}{1} = 1$ by L'Hôpital's rule. Furthermore $x_0 = a^0 = 1$ so demonstrating the validity of the theorem.

▶

7 The derivative of the transform

If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$

This is easily proved.

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k} \text{ and so } F'(z) = \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k}$$

$$= -\frac{1}{z}Z\{kx_k\}$$

and so $-zF'(z) = Z\{kx_k\}$

For example, the sequence $\{a^k\}$ has the Z transform $F(z) = \dfrac{z}{z-a}$ and so the sequence $\{ka^k\}$ has Z transform

$$Z\{kx_k\} = -zF'(z) = \ldots\ldots\ldots\ldots$$

18

$$\boxed{Z\{kx_k\} = \frac{az}{(z-a)^2}}$$

Because

$$-zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Notice that this is in agreement with the Table of transforms in Frame 8.

Next frame

Inverse transforms

19

If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}$$

There are many times when, given the Z transform of a sequence, it is not possible to immediately read off the sequence from the Table of transforms. Instead some manipulation may be required and, as with Laplace transforms, very often this involves using partial fractions.

Example

The sequence $\{x_k\}$ has Z transform $F(z) = \dfrac{z}{z^2 - 5z + 6}$. To find the inverse transform, and hence the sequence, we recognise that the denominator can be factorised and separated into partial fractions as

$$F(z) = \ldots\ldots\ldots\ldots$$

20

$$\boxed{F(z) = \frac{3}{z-3} - \frac{2}{z-2}}$$

Because

$$\begin{aligned}F(z) &= \frac{z}{z^2 - 5z + 6}\\ &= \frac{z}{(z-2)(z-3)}\\ &= \frac{A}{z-2} + \frac{B}{z-3}\\ &= \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators gives $z = A(z-3) + B(z-2)$, giving $A + B = 1$ and $-3A - 2B = 0$. From these two equations we find that $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $Z\{a^k\} = \frac{z}{z-a}$. Therefore if we write

$$\begin{aligned}F(z) &= \frac{3}{z-3} - \frac{2}{z-2}\\ &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2}\end{aligned}$$

so

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

21

$$\boxed{Z^{-1}F(z) = \{3^k - 2^k\}}$$

Because

$$\begin{aligned}F(z) &= \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2}\\ &= 3 \times z^{-1}Z\{3^k\} - 2 \times z^{-1}Z\{2^k\}\end{aligned}$$

and so

$$\begin{aligned}Z^{-1}F(z) &= 3 \times \{3^{k-1}\} - 2 \times \{2^{k-1}\} \text{ by the second shift theorem}\\ &= \{3^k\} - \{2^k\}\\ &= \{3^k - 2^k\} \text{ giving } x_k = 3^k - 2^k\end{aligned}$$

There is a simpler way of doing this without employing the second shift theorem. Recognising that z appears in the numerator of $F(z)$, we consider instead the partial fraction breakdown of $\frac{F(z)}{z}$

$$\frac{F(z)}{z} = \ldots\ldots\ldots\ldots$$

22

$$\boxed{\frac{1}{z-3}-\frac{1}{z-2}}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z}\times\frac{z}{z^2-5z+6}\\ &= \frac{1}{z^2-5z+6}\\ &= \frac{1}{(z-2)(z-3)}\\ &= \frac{A}{z-2}+\frac{B}{z-3}\\ &= \frac{A(z-3)+B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators gives $1 = A(z-3)+B(z-2)$, giving

[z]: $A+B=0$

[CT]: $-3A-2B=1$ with solution $A=-1$ and $B=1$. So that

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z-3}-\frac{1}{z-2} \text{ that is}\\ F(z) &= \frac{z}{z-3}-\frac{z}{z-2}\\ &= Z\{3^k\}-Z\{2^k\} \text{ and so}\\ Z^{-1}F(z) &= \{3^k\}-\{2^k\}\\ &= \{3^k-2^k\}\end{aligned}$$

Thus the use of the second shift theorem is avoided.

So try one yourself. The sequence $\{x_k\}$ has Z transform

$$F(z)=\frac{5z}{(z^2-4z+4)(z+2)}$$

therefore $\{x_k\} = \ldots\ldots\ldots\ldots$

23

$$\boxed{\{x_k\} = \left\{\frac{5k}{4} - \frac{5}{16} \times \left(2^k + (-2)^k\right)\right\}}$$

Because

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z} \times \frac{5z}{(z^2 - 4z + 4)(z+2)} \\
&= \frac{5}{(z-2)^2(z+2)} \\
&= \frac{A}{(z-2)^2} + \frac{B}{z-2} + \frac{C}{z+2} \\
&= \frac{A(z+2) + B(z-2)(z+2) + C(z-2)^2}{(z-2)^2(z+2)}
\end{aligned}$$

Equating numerators gives $5 = A(z+2) + B(z^2 - 4) + C(z^2 - 4z + 4)$, giving

$[z^2]$: $\quad B + C = 0$

$[z]$: $\quad A - 4C = 0$

$[CT]$: $\quad 2A - 4B + 4C = 5$

with solution $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2} \text{ giving}$$

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2} \text{ and so}$$

$$\begin{aligned}
Z^{-1}F(z) &= \frac{5}{8} \times \{k2^k\} - \frac{5}{16} \times \{2^k\} + \frac{5}{16} \times \left\{(-2)^k\right\} \\
&= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\}
\end{aligned}$$

Next frame

Recurrence relations

24

Sometimes adjacent terms of a sequence are related to each other. For example the terms of the sequence

$$\{x_k\} = \{2^k\}$$

are such that $x_{k+1} = 2^{k+1} = 2 \times 2^k = 2x_k$. That is

$$x_{k+1} = 2x_k$$

This equation holds true for all adjacent terms of the sequence – it *recurs* for all values of k. The equation is called a **linear, first-order, constant coefficient recurrence relation**. The order of the equation is given by the maximum shift between related terms – here it is 1. Clearly, the recurrence relation

$x_{k+2} - x_{k+1} - x_k = 1$ is of order ....

25

$$\boxed{2}$$

Because

The maximum shift between terms in the relation is 2 – that is from k to $k+2$.

Initial terms

A recurrence relation can be used to generate the terms of a sequence provided initial terms are given – equal in number to the order of the equation. For example, given the sequence $\{x_k\}$ where $x_{k+1} = 3x_k$ with the initial term $x_0 = 2$ generates the sequence of terms

$$\{x_k\} = \{2, \ldots, \ldots, \ldots, \ldots\}$$

26

$$\boxed{\{x_k\} = \{2, 6, 18, 54, \ldots\}}$$

Because

Since $x_{k+1} = 3x_k$ where $x_0 = 2$ then

$$x_1 = 3x_0 = 3 \times 2 = 6$$
$$x_2 = 3x_1 = 3 \times 6 = 18$$
$$x_3 = 3x_2 = 3 \times 18 = 54$$

Similarly, if another sequence has terms that satisfy the second-order recurrence relation

$$x_{k+2} - 3x_{k+1} + 2x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = 1$$

then the first five terms of the sequence are

$$\{x_k\} = \{0, 1, \ldots, \ldots, \ldots, \ldots\}$$

27

$$\boxed{\{x_k\} = \{0, 1, 4, 11, 26, \ldots\}}$$

Because

Since $x_{k+2} - 3x_{k+1} + 2x_k = 1$ where $x_0 = 0$ and $x_1 = 1$ then

$$x_2 - 3x_1 + 2x_0 = 1 \text{ that is } x_2 - 3 \times 1 + 2 \times 0 = 1 \text{ and so } x_2 = 4$$
$$x_3 - 3x_2 + 2x_1 = 1 \text{ that is } x_3 - 3 \times 4 + 2 \times 1 = 1 \text{ and so } x_3 = 11$$
$$x_4 - 3x_3 + 2x_2 = 1 \text{ that is } x_4 - 3 \times 11 + 2 \times 4 = 1 \text{ and so } x_4 = 26$$

Try another yourself.

The sequence $\{x_k\}$ has terms that satisfy the second-order recurrence relation

$$x_{k+2} - x_k = 1 \text{ where } x_0 = 0 \text{ and } x_1 = -1$$

The first six terms of this sequence are

$$\{x_k\} = \{0, -1, \ldots, \ldots, \ldots, \ldots, \ldots\}$$

28

$$\boxed{\{x_k\} = \{0, \; -1, \; 1, \; 0, \; 2, \; 1, \; \ldots\}}$$

Because

Since $x_{k+2} - x_k = 1$ where $x_0 = 0$ and $x_1 = -1$ then

$x_2 - x_0 = 1$ that is $x_2 - 0 = 1$ and so $x_2 = 1$

$x_3 - x_1 = 1$ that is $x_3 + 1 = 1$ and so $x_3 = 0$

$x_4 - x_2 = 1$ that is $x_4 - 1 = 1$ and so $x_4 = 2$

$x_5 - x_3 = 1$ that is $x_5 - 0 = 1$ and so $x_5 = 1$

Therefore $\{x_k\} = \{0, \; -1, \; 1, \; 0, \; 2, \; 1, \; \ldots\}$

Next frame

Solving the recurrence relation

29

If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the Z transform where $Z\{x_k\} = F(z)$. This is referred to as *solving the recurrence relation*. For example, solve the recurrence relation

$x_{k+2} - 3x_{k+1} + 2x_k = 1$ where $x_0 = 0$ and $x_1 = 1$

Because this recurrence relation is true for all values of k it can itself be used to form a sequence $\{y_k\}$, namely

$$\{y_k\} = \{x_{k+2} - 3x_{k+1} + 2x_k\} = \{1\}$$

Now, taking the Z transform of both sides of this equation gives

$Z\{y_k\} = Z\{x_{k+2} - 3x_{k+1} + 2x_k\} = Z\{1\}$ that is

$Z\{x_{k+2}\} - 3Z\{x_{k+1}\} + 2\{x_k\} = Z\{1\}$

Using the first shift theorem and $Z\{x_k\} = F(z)$ this then becomes

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - 3(zF(z) - zx_0) + 2F(z) = \frac{z}{z-1}$$

Collecting like terms and substituting for the initial terms $x_0 = 0$ and $x_1 = 1$ gives

$(z^2 - 3z + 2)F(z) - z = \dfrac{z}{z-1}$ so $(z^2 - 3z + 2)F(z) = z + \dfrac{z}{z-1} = \dfrac{z^2}{z-1}$

That is $F(z) = \dfrac{z^2}{(z-1)(z^2 - 3z + 2)} = \dfrac{z^2}{(z-1)^2(z-2)}$

and so $\dfrac{F(z)}{z} = \dfrac{z}{(z-1)^2(z-2)}$

This has the partial fraction breakdown

$$\frac{F(z)}{z} = \frac{\cdots\cdots}{(z-1)^2} \cdots \frac{\cdots\cdots}{z-1} \cdots \frac{\cdots\cdots}{z-2}$$

30

$$\boxed{\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{z}{(z-1)^2(z-2)}\\ &= \frac{A}{(z-1)^2} + \frac{B}{z-1} + \frac{C}{z-2}\\ &= \frac{A(z-2) + B(z-1)(z-2) + C(z-1)^2}{(z-1)^2(z-2)}\end{aligned}$$

and so

$z = A(z-2) + B(z-1)(z-2) + C(z-1)^2$ giving

$[z^2]$: $B + C = 0$

$[z^1]$: $A - 3B - 2C = 1$

[CT]: $-2A + 2B + C = 0$

with solution $A = -1$, $B = -2$ and $C = 2$

Therefore

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2}$$

Taking the inverse Z transform of $F(z)$ yields the sequence

$$Z^{-1}F(z) = \ldots\ldots\ldots\ldots$$

31

$$\boxed{Z^{-1}F(z) = \{-k - 2 + 2^{k+1}\}}$$

Because

$$\frac{F(z)}{z} = -\frac{1}{(z-1)^2} - \frac{2}{z-1} + \frac{2}{z-2} \text{ and so}$$

$$F(z) = -\frac{z}{(z-1)^2} - \frac{2z}{z-1} + \frac{2z}{z-2}$$

Therefore

$$\begin{aligned} Z^{-1}F(z) &= -Z^{-1}\left(\frac{z}{(z-1)^2}\right) - 2Z^{-1}\left(\frac{z}{z-1}\right) + 2Z^{-1}\left(\frac{z}{z-2}\right) \\ &= \{-k - 2x_k + 2(2^k)\} \\ &= \{-k - 2 + 2^{k+1}\} \text{ since } x_k = 1 \end{aligned}$$

Indeed, $\{x_k\} = \{-k - 2 + 2^{k+1}\}$ is the solution to the recurrence relation as can be seen by substituting back

$$\begin{aligned} &x_{k+2} - 3x_{k+1} + 2x_k \\ &= \left(-[k+2] - 2 + 2^{[k+2]+1}\right) - 3\left(-[k+1] - 2 + 2^{[k+1]+1}\right) \\ &\quad + 2(-k - 2 + 2^{k+1}) \\ &= (-k - 4 + 8 \times 2^k) - 3(-k - 3 + 4 \times 2^k) + 2(-k - 2 + 2 \times 2^k) \\ &= -k - 4 + 8 \times 2^k + 3k + 9 - 12 \times 2^k - 2k - 4 + 4 \times 2^k \\ &= 1 \end{aligned}$$

Try one yourself.

The solution of the second-order recurrence relation

$x_{k+2} - x_k = 1$ where $x_0 = 0$ and $x_1 = -1$ is $x_k = \ldots\ldots\ldots\ldots$

32

$$\boxed{x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}}$$

Because

Taking the Z transform of the recurrence relation gives

$Z\{x_{k+2} - x_k\} = Z\{1\}$. That is, $Z\{x_{k+2}\} - Z\{x_k\} = Z\{1\}$ so that

$$(z^2F(z) - z^2x_0 - zx_1) - F(z) = \frac{z}{z-1}.$$

Substituting for $x_0 = 0$ and $x_1 = -1$ gives

$$F(z) = \ldots\ldots\ldots\ldots$$

33

$$F(z) = \frac{-z+2}{(z+1)(z-1)^2}$$

Because

$$\left(z^2F(z) - z^2x_0 - zx_1\right) - F(z) = \frac{z}{z-1} \text{ where } x_0 = 0 \text{ and } x_1 = -1 \text{ giving}$$

$$(z^2 - 1)F(z) + z = \frac{z}{z-1} \text{ so}$$

$$F(z) = \frac{z}{(z^2-1)(z-1)} - \frac{z}{(z^2-1)} \text{ so}$$

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{(z+1)(z-1)^2} - \frac{1}{(z+1)(z-1)}\\ &= \frac{1-(z-1)}{(z+1)(z-1)^2}\\ &= \frac{-z+2}{(z+1)(z-1)^2}\end{aligned}$$

Separating into partial fractions gives

$$\frac{F(z)}{z} = \ldots\ldots\ldots\ldots$$

34

$$\frac{F(z)}{z} = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Because

$$\begin{aligned}\frac{F(z)}{z} &= \frac{-z+2}{(z+1)(z-1)^2}\\ &= \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}\\ &= \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}\end{aligned}$$

Equating numerators and comparing coefficients of powers of z gives

$[z^2]$: $A + B = 0$

$[z]$: $-2A + C = -1$

[CT]: $A - B + C = 2$ with solution $A = 3/4$, $B = -3/4$ and $C = 1/2$

so that $F(z) = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$

By inverting the transform we find that

$$x_k = \ldots\ldots\ldots\ldots$$

35

$$\boxed{x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}}$$

Because

$$F(z) = \frac{3}{4}\frac{z}{z+1} - \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

and

$$Z^{-1}\left\{\frac{z}{z+1}\right\} = \left\{(-1)^k\right\} \text{ so } Z^{-1}\left\{(3/4)\frac{z}{z+1}\right\} = (3/4)\left\{(-1)^k\right\}$$
$$Z^{-1}\left\{\frac{z}{z-1}\right\} = \{1^k\} \text{ so } Z^{-1}\left\{(-3/4)\frac{z}{z-1}\right\} = (-3/4)\{1^k\}$$
$$Z^{-1}\left\{\frac{z}{(z-1)^2}\right\} = \{k\} \text{ so } Z^{-1}\left\{(1/2)\frac{z}{(z-1)^2}\right\} = (1/2)\{k\}$$

Therefore $\{x_k\} = \left\{(3/4)(-1)^k - (3/4) + (k/2)\right\}$

so that $x_k = \begin{cases} k/2 & k \text{ even} \\ (k-3)/2 & k \text{ odd} \end{cases}$

Next frame

Sampling

36

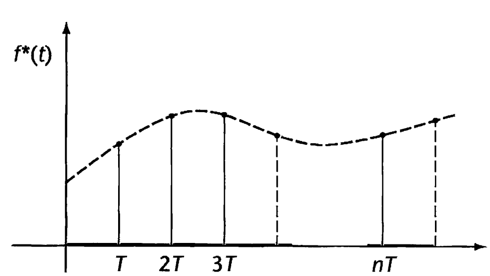
If a continuous function $f(t)$ of time t progresses from $t = 0$ onwards and is measured at every time interval T then what will result is the sequence of values

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\}$$

A new, piecewise continuous function $f^*(t)$ can then be created from the sequence of sampled values such that

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = kT \\ 0 & \text{otherwise} \end{cases}$$

The graph of this new function consists of a series of spikes at the regular intervals $t = kT$



▶

This function can alternatively be described in terms of the delta function $\delta(t)$ as

$$f^*(t) = f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + f(3T)\delta(t-3T) + \ldots$$
$$= \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

The Laplace transform of $f^*(t)$ is then given as

$$F^*(s) = L\{f^*(t)\}$$
$$= \int_0^{\infty} \{f(0)\delta(t) + f(T)\delta(t-T) + f(2T)\delta(t-2T) + \ldots\}e^{-st}\,dt$$
$$= f(0) + f(T)e^{-sT} + f(2T)e^{-2sT} + f(3T)e^{-3sT} + \ldots$$
$$= \sum_{k=0}^{\infty} f(kT)e^{-ksT}$$

Define a new variable $z = e^{sT}$ and we see that

$$L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$$

which is the Z transform of the sequence$\{f(kT)\}$.

Example 1

The function $f(t) = e^{-at}$ is sampled every interval of T.

The Z transform of the sampled function is then …………

37

$$\boxed{F(z) = \frac{z}{z - e^{-aT}}}$$

Because

Defining $f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT) = \sum_{k=0}^{\infty} e^{-akT}\delta(t-kT)$ then the Laplace transform of $f^*(t)$ is given as

$$F^*(s) = \sum_{k=0}^{\infty} e^{-kaT}e^{-ksT}$$

This means that the Z transform of $\{f(kT)\}$ is

$$F(z) = \sum_{k=0}^{\infty} \frac{e^{-kaT}}{z^k} = \frac{1}{1 - \dfrac{e^{-aT}}{z}} = \frac{z}{z - e^{-aT}}$$

Notice that this agrees with the Z transform of the sequence $\{b^k\}$ $\left(\text{which is } \dfrac{z}{z-b}\right)$ when b is replaced by e^{-aT}.

Try another.

Example 2

The function $f(t) = t$ is sampled every interval of T.

The Z transform of the sampled function is then …………

38

$$F(z) = \frac{Tz}{(z-1)^2}$$

Because

The Z transform of $\{f(kT)\}$ is $F(z) = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k}$. Here $f(kT) = kT$ and so

$$\begin{aligned}
F(z) &= \sum_{k=0}^{\infty} \frac{kT}{z^k} \\
&= T\left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots\right) \\
&= \frac{T}{z}(1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \dots) \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}(1 + z^{-1} + z^{-2} + z^{-3} + \dots\} \\
&= -Tz\frac{\mathrm{d}}{\mathrm{d}z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{T}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{Tz}{(z-1)^2}
\end{aligned}$$

Example 3

The function $f(t) = \cos t$ is sampled every interval of T.

The Z transform of the sampled function is then ....

39

$$F(z) = \frac{z(z - \cos T)}{z^2 - 2\cos T + 1}$$

Because

$f(t) = \cos t = \dfrac{e^{jT} + e^{-jT}}{2}$ and the Z transform of $\{e^{-kaT}\}$ is $F(z) = \dfrac{z}{z - e^{-aT}}$.

Therefore the Z transform of $\dfrac{e^{jT} + e^{-jT}}{2}$ is

$$\begin{aligned}
\frac{1}{2}\left(\frac{z}{z - e^{-jT}} + \frac{z}{z - e^{jT}}\right) &= \frac{1}{2}\left(\frac{z(z - e^{jT}) + z(z - e^{-jT})}{(z - e^{-jT})(z - e^{jT})}\right) \\
&= \frac{1}{2}\left(\frac{2z^2 - z(e^{jT} + e^{-jT})}{z^2 - [e^{jT} + e^{-jT}]z + 1}\right) \\
&= \frac{z(z - \cos T)}{z^2 - 2z\cos T + 1}
\end{aligned}$$

▶

And that is the end of the Programme on Z transforms. All that remain are the **Revision summary** and the **Can You?** checklist. Read through these closely and make sure that you understand all the workings of this Programme. Then try the **Test exercise**; there is no need to hurry, take your time and work through the questions carefully. The **Further problems** then provide a valuable collection of additional exercises for you to try.

40

Revision summary 5

1 *Sequences*

The sequence $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ is represented by the notation $\{x_k\}_{-\infty}^{\infty}$. The sequence $\{x_k\}_0^{\infty}$ is called a causal sequence and is denoted simply by $\{x_k\}$.

2 *Z transform*

The Z transform of the causal sequence $\{x_k\}$ is

$$Z\{x_k\} = \sum_{k=0}^{\infty}\left(\frac{x_k}{z^k}\right) = F(z)$$ where the value of z is chosen to ensure that the sum converges.

$\{x_k\}$ and $Z\{x_k\}$ form a Z transform pair.

3 *Table of Z transforms*

Sequence	**Transform $F(z)$**	**Permitted values of z**
$\{\delta_k\} = \{1, 0, 0, \ldots\}$	1	All values of z
$\{x_k\} = \{1, 1, 1, \ldots\}$	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\dfrac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\dfrac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\dfrac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

4 *Linearity*

The Z transform is a linear transform. That is, if a and b are constants then

$$Z(a\{x_k\} + b\{y_k\}) = aZ\{x_k\} + bZ\{y_k\}.$$

▶

5 *First shift theorem* (shifting to the left)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k+m}\} = z^m F(z) - \left[z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}\right]$$

the Z transform of the sequence that has been shifted by m places to the left.

6 *Second shift theorem* (shifting to the right)
If $Z\{x_k\} = F(z)$ then

$$Z\{x_{k-m}\} = z^{-m} F(z)$$

the Z transform of the sequence that has been shifted by m places to the right.

7 *Translation*
If the sequence $\{x_k\}$ has the Z transform $Z\{x_k\} = F(z)$ then the sequence $\{a^k x_k\}$ has the Z transform $Z\{a^k x_k\} = F(a^{-1}z)$.

8 *Final value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\underset{k\to\infty}{Lim}\, x_k = \underset{z\to 1}{Lim}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \text{ provided that } \underset{k\to\infty}{Lim}\, x_k \text{ exists.}$$

9 *The initial value theorem*
For the sequence $\{x_k\}$ with Z transform $F(z)$

$$x_0 = \underset{z\to\infty}{Lim}\{F(z)\}.$$

10 *The derivative of the transform*
If $Z\{x_k\} = F(z)$ then $-zF'(z) = Z\{kx_k\}$.

11 *Inverse transformations*
If the sequence $\{x_k\}$ has Z transform $Z\{x_k\} = F(z)$, the inverse transform is defined as

$$Z^{-1}F(z) = \{x_k\}.$$

12 *Recurrence relations*
A recurrence relation expresses the relationship that adjacent terms of a series hold to each other. The order of the equation is given by the maximum shift between related terms.

Initial terms
A recurrence relation can be used to generate the terms of a sequence provided initial terms are given – equal in number to the order of the equation.

Solving the recurrence relation
If a sequence $\{x_k\}$ satisfies a recurrence relation with given initial conditions then the general term of the sequence can be found by using the Z transform where $Z\{x_k\} = F(z)$. This is referred to as *solving the recurrence relation.*

▶

13 *Sampling*

If a continuous function $f(t)$ is sampled at equal intervals, the resulting sequence has a Z transform that is related to the Laplace transform of the piecewise function created from the sequence of sample values.

$$L\{f^*(t)\} = \sum_{k=0}^{\infty} f(kT)z^{-k} = \sum_{k=0}^{\infty} \frac{f(kT)}{z^k} = Z\{f(kT)\}$$

where

$$\{f(kT)\} = \{f(0), f(T), f(2T), f(3T), \ldots\},$$

$$f^*(t) = \begin{cases} f(kT) & \text{if } t = k \\ 0 & \text{otherwise} \end{cases}$$

and

$$z = e^{sT}.$$

Can You?

41 Checklist 5

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: **Frames**

- Define the Z transform of a sequence and derive transforms of specified sequences? 1 to 7

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Make reference to a table of standard Z transforms? 8

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Recognise the Z transform as being a linear transform and so obtain the transform of linear combinations of standard sequences? 9 to 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Apply the first and second shift theorems, the translation theorem, the initial and final value theorems and the derivative theorem? 11 to 18

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use partial fractions to derive the inverse transforms? 19 to 23

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve linear, constant coefficient recurrence relations? 24 to 35

Yes ☐ ☐ ☐ ☐ ☐ *No*

- Demonstrate the relationship between the Laplace transform and the Z transform? 36 to 39

Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 5

42

1 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-1)^k$.

2 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = 4k - 2a^k$.

3 Find the Z transform of the causal sequences:

(a) $\{k - 3\}$

(b) $\{5^{k+2}\}$

4 Find the inverse Z transformation of

$$F(z) = \frac{z^2(z-3)}{(z^2 - 2z + 1)(z-2)}.$$

5 Solve the recurrence relation

$$x_{k+2} - 4x_{k+1} + 4x_k = 3 \text{ where } x_0 = 1 \text{ and } x_1 = 0.$$

6 The function $f(t) = \sin t$ is sampled at equal intervals of $t = T$. Find the Z transform of the resulting sequence of values.

Further problems 5

43

1 Find the Z transform of the causal sequence $\{x_k\}$ where $x_k = (-a)^k$ where $a > 0$.

2 Solve each of the following recurrence relations.

(a) $x_{k+2} + 5x_{k+1} + 6x_k = 1$ where $x_0 = 0$ and $x_1 = 1$

(b) $3x_{k+2} - 7x_{k+1} + 2x_k = k$ where $x_0 = 1$ and $x_1 = 0$

(c) $x_{k+2} - 9x_k = 2k$ where $x_0 = 1$ and $x_1 = 1$.

3 Given that $y_{k+1} = v_k$ and $v_{k+1} = w_k$ where $w_k = x_k - y_k$, show that $y_{k+2} + y_k = x_k$ and solve for y_k when $\{x_k\} = \{\delta_k\}$, the unit impulse sequence where $y_0 = 0$, $y_1 = 1$.

▶

4 If

$$p_{k+1} = q_k$$
$$q_{k+1} = r_k$$
$r_k = x_k - \alpha q_k - \beta p_k$ where α and β are constants, show that
$$p_{k+2} + \alpha p_{k+1} + \beta p_k = x_k$$

Solve this recurrence relation when $p_0 = 1$, $p_1 = 0$ for

(a) $\alpha = 4$, $\beta = 4$ and $\{x_k\} = \{\delta_k\}$, the unit impulse sequence

(b) $\alpha = 4$, $\beta = 4$ and $\{x_k\} = \{u_k\}$ the unit step sequence.

5 Find the Z transform of each of the following sequences.

(a) $\{1, 0, 1, 0, 1, 0, \ldots\}$

(b) $\{0, 1, 0, 1, 0, 1, \ldots\}$

(c) $\{1, 0, 1, 1, 0, 0, 0, 1\}$

(d) $\{1, 1, 1, 0, 0, 0, 1, 1\}$

(e) $\{0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 1\}$

(f) $\{1, 1, 0, 0, 0, 1, 1\}$

Note that the last four of these are finite sequences.

6 Find the inverse transform of

(a) $F(z) = \dfrac{z}{(z+1)(z+2)(z+3)}$

(b) $F(z) = \dfrac{z^2}{(z+1)(z+2)(z+3)}$

(c) $F(z) = \dfrac{z(3z+1)}{(z-2)(z-3)}$

(d) $F(z) = \dfrac{z^2}{2 - 3z + z^2}$.

7 Given

$$F(z) = \frac{3z^2}{z^2 - z + 1}$$

show that

$$Z^{-1}F(z) = \{3, 3, -3, -3, \ldots\}.$$

Hint: Use long division on $F(z)$.

8 Given

$$F(z) = \left(1 + \frac{2}{z}\right)^{-3}$$

show that

$$Z^{-1}F(z) = \{1, -6, 24, -48, \ldots\}.$$

Hint: Use the binomial theorem on $F(z)$.

▶

9 Find the final value of the sequence $\{x_k\}$ with Z transform

$$F(z) = \frac{4z^2 - z}{2z^2 - 3z + 1}.$$

10 What is the initial value of the sequence whose Z transform is given by

$$F(z) = \frac{2z^2 - z + 1}{5 - 3z - 7z^2}?$$

11 Given the sequence of n terms $\{x_k\}$ for $0 \le k \le n - 1$ with Z transform $F_n(z)$, show that the Z transform of the sequence formed by continually repeating the terms $\{x_k\}$ is given as

$$F(z) = \frac{F_n(z)}{1 - z^{-n}}.$$

12 Using the result of Question 11, show that the Z transform of the sequence obtained by continually repeating the three term sequence $\{1, 0, -1\}$ is

$$F(z) = \frac{z^2}{z^2 + 1}.$$

13 Find the Z transforms of the sequence of values obtained when $f(t)$ is sampled at regular intervals of $t = T$ where

(a) $f(t) = \sinh t$

(b) $f(t) = \cosh at$

(c) $f(t) = e^{-at} \cosh bt$.